

Direct relation of linearized supergravity to superconformal formulation

Yutaka Sakamura^{*}

*KEK Theory Center, Institute of Particle and Nuclear Studies, KEK,
Tsukuba, Ibaraki 305-0801, Japan*

*Department of Particles and Nuclear Physics,
The Graduate University for Advanced Studies (Sokendai),
Tsukuba, Ibaraki 305-0801, Japan*

Abstract

We modify the four-dimensional $N = 1$ linearized supergravity in a way that components in each superfield are completely identified with fields in the full superconformal formulation. This identification makes it possible to use both formulations of supergravity in a complementary manner. It also provides a basis for an extension to higher-dimensional supergravities.

^{*}e-mail address: sakamura@post.kek.jp

1 Introduction

The superconformal formulation of supergravity (SUGRA) is a powerful and systematic method for constructing various SUGRA actions [1, 2, 3, 4]. Most of the known off-shell SUGRA actions are reproduced by this formulation. It has also been extended to the five-dimensional (5D) case [5, 6], which is useful to discuss the brane-world scenario based on general 5D SUGRA. Although the actions are obtained in a systematic way, their explicit expressions are lengthy and awkward due to a number of auxiliary fields. Especially the couplings between the matter and the SUGRA fields (*i.e.*, the vierbein, the gravitino, etc.) are complicated.

Linearized supergravity [7, 8] is easier to deal with because it is described in terms of superfields on the ordinary superspace. It is powerful for some calculations because the ordinary superfield techniques are applicable just as in the global supersymmetry (SUSY) case. An extension to 5D case for the minimal set-up was done in Ref. [9], and it makes it possible to calculate the SUGRA loop contributions in the 5D brane-world models [10], keeping the $N = 1$ SUSY off-shell structure. On the other hand, we cannot use this formalism for calculations beyond the linearized order in the SUGRA fields. The full SUGRA formulation, such as the superconformal formulation, is necessary for them.

Therefore it will be useful to combine the two formulations in a complementary manner. In fact, it is pointed out in Ref. [7, 8] that the linearized SUGRA transformations contain some of the superconformal transformations at the linearized level. Although both formulations are self-consistent, an explicit relation between them has not been provided so far. This is the main obstacle to the complementary use of them.

In this paper, we will modify the linearized SUGRA formulation and provide a complete identification of component fields in each superfield with fields in the superconformal formulation developed in Ref. [4]. This identification also provides a basis for an extension to higher-dimensional SUGRA.

The paper is organized as follows. In Sec. 2, we consider superfield transformations which are identified with the linearized superconformal transformations. In Sec. 3, we translate such transformation laws into those for component fields, and identify the fields and the transformation parameters with those in the superconformal formulation of Ref. [4]. In Sec. 4, we construct the invariant action formulae in terms of the superfields, which are consistent with those in Ref. [4]. Sec. 5 is devoted to the summary. In Appendix A, we provide explicit component expressions of some superfields in the text, and in Appendix B, we collect the invariant action formulae in Ref. [4] in our notations.

2 Superfield transformations

In this section, we consider the transformation laws of $N = 1$ superfields. We assume that the background geometry is a flat 4D Minkowski spacetime. Basically we use the spinor notations of Ref. [11], except for the metric and the spinor derivatives. We take the background metric as $\eta_{\mu\nu} = (1, -1, -1, -1)$ so as to match it to that of Ref. [4], and we define the spinor derivatives D_α and $\bar{D}_{\dot{\alpha}}$ as

$$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} - i (\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i (\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu, \quad (2.1)$$

which satisfy $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$.¹

2.1 Super-diffeomorphism

We begin with a brief review of the formulation developed in Ref. [8]. (A compact review is also provided in Ref. [9].) First we consider the super-diffeomorphism transformation acting on a chiral superfield Φ . It is expressed as

$$\delta\Phi = \Lambda^\alpha D_\alpha \Phi + \Lambda^\mu \partial_\mu \Phi, \quad (2.2)$$

where Λ^α and Λ^μ are a spinor and a vector superfields, respectively. We require that this transformation δ preserves the chiral condition,

$$\bar{D}_{\dot{\alpha}} \Phi = 0. \quad (2.3)$$

Then we obtain

$$\bar{D}_{\dot{\alpha}} \delta\Phi = \bar{D}_{\dot{\alpha}} \Lambda^\alpha D_\alpha \Phi + (-2i\Lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu + \bar{D}_{\dot{\alpha}} \Lambda^\mu) \partial_\mu \Phi = 0. \quad (2.4)$$

We have used (2.3). Thus we find that

$$\bar{D}_{\dot{\alpha}} \Lambda^\alpha = 0, \quad -2i\Lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu + \bar{D}_{\dot{\alpha}} \Lambda^\mu = 0. \quad (2.5)$$

The most general solution to these conditions can be parametrized by

$$\Lambda^\alpha = -\frac{1}{4} \bar{D}^2 L^\alpha, \quad \Lambda^\mu = -i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha - \Omega^\mu, \quad (2.6)$$

¹ In the notation of Ref. [11], the spinor derivatives satisfy $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$. Then, the (global) SUSY generators Q_α and $\bar{Q}_{\dot{\alpha}}$, which anticommute with D_α and $\bar{D}_{\dot{\alpha}}$, satisfy $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$. This leads to the SUSY algebra with an opposite sign to the usual one. (See Chapter IV of Ref. [11].)

where L^α is a general complex spinor superfield and Ω^μ is a chiral superfield. In terms of L^α and Ω^μ , the transformation of a chiral superfield Φ is rewritten as²

$$\begin{aligned}\delta\Phi &= -\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha \Phi - (i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu \Phi \\ &= -\frac{1}{4}\bar{D}^2 (L^\alpha D_\alpha \Phi) - \Omega^\mu \partial_\mu \Phi.\end{aligned}\tag{2.7}$$

Similarly, we can find the transformation acting on an anti-chiral superfield $\bar{\Phi}$ as

$$\begin{aligned}\delta\bar{\Phi} &= -\frac{1}{4}D^2 \bar{L}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi} - (i\sigma_{\alpha\dot{\alpha}}^\mu D^\alpha \bar{L}^{\dot{\alpha}} + \bar{\Omega}^\mu) \partial_\mu \bar{\Phi} \\ &= -\frac{1}{4}D^2 (\bar{L}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\Phi}) - \bar{\Omega}^\mu \partial_\mu \bar{\Phi}.\end{aligned}\tag{2.8}$$

This preserves the anti-chiral condition $D_\alpha \bar{\Phi} = 0$.

Next we consider the transformation of a product of a chiral and an anti-chiral superfields Φ_1 and $\bar{\Phi}_2$. In order to define the transformation acting on the product that is consistent with (2.7) and (2.8), we introduce a real superfield U^μ that transforms inhomogeneously as

$$\delta U^\mu = \frac{1}{2}\sigma_{\alpha\dot{\alpha}}^\mu (\bar{D}^{\dot{\alpha}} L^\alpha - D^\alpha \bar{L}^{\dot{\alpha}}) - \frac{i}{2}(\Omega^\mu - \bar{\Omega}^\mu),\tag{2.9}$$

and insert it into the product as

$$\bar{\Phi}_2 \Phi_1 \rightarrow \mathcal{V}(\bar{\Phi}_2 \Phi_1) \equiv \bar{\Phi}_2 \left(1 + iU^\mu \overset{\leftrightarrow}{\partial}_\mu\right) \Phi_1,\tag{2.10}$$

where $A \overset{\leftrightarrow}{\partial}_\mu B \equiv A \partial_\mu B - (\partial_\mu A) B$. As we will see in the next section, U^μ contains the SUGRA fields. The inserted product $\mathcal{V}(\bar{\Phi}_2 \Phi_1)$ transforms as

$$\begin{aligned}\delta\mathcal{V}(\bar{\Phi}_2 \Phi_1) &= \delta\bar{\Phi}_2 \Phi_1 + i\bar{\Phi}_2 \delta U^\mu \overset{\leftrightarrow}{\partial}_\mu \Phi_1 + \bar{\Phi}_2 \delta\Phi_1 \\ &= \left\{ -\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - \frac{1}{4}D^2 \bar{L}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} - \frac{i}{2}\sigma_{\alpha\dot{\alpha}}^\mu (\bar{D}^{\dot{\alpha}} L^\alpha + D^\alpha \bar{L}^{\dot{\alpha}}) \partial_\mu - \frac{1}{2}(\Omega^\mu + \bar{\Omega}^\mu) \partial_\mu \right\} (\bar{\Phi}_2 \Phi_1) \\ &= \left\{ -\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - \frac{1}{2}(i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu + \text{h.c.} \right\} \mathcal{V}(\bar{\Phi}_2 \Phi_1).\end{aligned}\tag{2.11}$$

Here and henceforth, we neglect the U^μ -dependent terms in the right-hand sides of the transformation laws because they are irrelevant to an invariance of the action at the linearized order in U^μ . A general superfield has the same transformation law as (2.11). Note that this law preserves the reality condition.

² We keep both L^α and Ω^μ as the transformation parameters while the latter is set to zero in Ref. [9]. In our formulation, the degrees of freedom in Ω^μ will be eliminated by the constraints (3.5) or absorbed into $\tilde{\xi}_I$ in (3.8).

P	M	Q	
translation	local Lorentz	SUSY	
D	$U(1)_A$	S	K
dilatation	R-symmetry	conformal SUSY	conformal boost

Table I: 4D $N = 1$ superconformal transformations

2.2 Superconformal transformations

It is mentioned in Ref. [8] that the transformation δ discussed in the previous subsection contain some of the superconformal transformations in addition to the super-Poincaré ones. (The 4D $N = 1$ superconformal transformations are listed in Table I.) However it is unclear how those transformations are related to those of Ref. [4]. For example, the δ -transformation law of the conformal compensator superfield is essentially different from that of a matter chiral superfield [9]. On the other hand, they transform in the same way in the superconformal formulation [4], except for the Weyl weight w and the chiral weight n , which are the charges of D and $U(1)_A$.

In order to incorporate these weights explicitly into the superfield transformations, we modify the transformation acting on a general superfield Ψ as

$$\delta_{\text{sc}}\Psi \equiv \delta\Psi + (w + n)\Lambda\Psi + (w - n)\bar{\Lambda}\bar{\Psi}, \quad (2.12)$$

where w and n are the Weyl and the chiral weights of Ψ ,³ and Λ is a chiral superfield to be determined later. (See eq.(3.13).) This modified transformation δ_{sc} preserves the (anti-)chiral condition or the reality condition, and satisfy the Leibniz rule since both weights are additive quantum numbers. Because $w = n$ and $w = -n$ for a chiral and an anti-chiral superfields Φ and $\bar{\Phi}$, (2.7) and (2.8) are modified as

$$\begin{aligned} \delta_{\text{sc}}\Phi &= \left\{ -\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - (i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu + 2w\Lambda \right\} \Phi, \\ \delta_{\text{sc}}\bar{\Phi} &= \left\{ -\frac{1}{4}D^2 \bar{L}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} - (i\sigma_{\alpha\dot{\alpha}}^\mu D^\alpha \bar{L}^{\dot{\alpha}} + \bar{\Omega}^\mu) \partial_\mu + 2w\bar{\Lambda} \right\} \bar{\Phi}. \end{aligned} \quad (2.13)$$

A real general superfield V (*i.e.*, $n = 0$) transforms as

$$\delta_{\text{sc}}V = \left\{ -\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - \frac{1}{2}(i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu + w\Lambda + \text{h.c.} \right\} V. \quad (2.14)$$

³ The Weyl and the chiral weights of a superfield denote those of the lowest component in the superfield.

The transformation of U^μ does not change from (2.9),

$$\delta_{\text{sc}} U^\mu = \frac{1}{2} \sigma_{\alpha\dot{\alpha}}^\mu (\bar{D}^{\dot{\alpha}} L^\alpha - D^\alpha \bar{L}^{\dot{\alpha}}) - \frac{i}{2} (\Omega^\mu - \bar{\Omega}^\mu). \quad (2.15)$$

The manner of inserting the connection superfield U^μ in (2.10) is generalized to a product of arbitrary superfields $\Psi_1, \Psi_2, \dots, \Psi_n$ as

$$\Psi_1 \Psi_2 \cdots \Psi_n \rightarrow \mathcal{V}(\Psi_1 \Psi_2 \cdots \Psi_n) \equiv \left(1 + i U^\mu \hat{\partial}_\mu\right) (\Psi_1 \Psi_2 \cdots \Psi_n), \quad (2.16)$$

where

$$\hat{\partial}_\mu \equiv \begin{cases} \partial_\mu & \text{(on chiral superfields)} \\ 0 & \text{(on general superfields)} \\ -\partial_\mu & \text{(on anti-chiral superfields)} \end{cases} \quad (2.17)$$

Then the transformation δ_{sc} acts on this product as

$$\begin{aligned} \delta_{\text{sc}} \mathcal{V}(\Psi_1 \Psi_2 \cdots \Psi_n) = & \left\{ -\frac{1}{4} \bar{D}^2 L^\alpha D_\alpha - \frac{1}{2} (i \sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu + \text{h.c.} \right. \\ & \left. + (w + n) \Lambda + (w - n) \bar{\Lambda} \right\} \mathcal{V}(\Psi_1 \Psi_2 \cdots \Psi_n), \end{aligned} \quad (2.18)$$

where w and n are the Weyl and the chiral weights for the product $\Psi_1 \Psi_2 \cdots \Psi_n$. Notice that \mathcal{V} defined in (2.16) is regarded as an embedding into a general superfield.

3 Identification of component fields

3.1 SUGRA multiplet and transformation parameters

Now we will see the transformation laws for component fields. First we consider the connection superfields U_μ , whose components are defined as

$$U_\mu = u_\mu + \theta \chi_\mu^U + \bar{\theta} \bar{\chi}_\mu^U + \theta^2 a_\mu + \bar{\theta}^2 \bar{a}_\mu + (\theta \sigma^\nu \bar{\theta}) \tilde{e}_{\nu\mu} + \bar{\theta}^2 (\theta \tilde{\psi}_\mu) + \theta^2 (\bar{\theta} \tilde{\bar{\psi}}_\mu) + \theta^2 \bar{\theta}^2 d_\mu, \quad (3.1)$$

where u_μ , $\tilde{e}_{\nu\mu}$ and d_μ are real. Note that $\tilde{e}_{\nu\mu}$ is neither symmetric nor anti-symmetric for the indices. The transformation parameter superfields are expanded as

$$\begin{aligned} L_\alpha = & l_\alpha + \theta_\alpha v + (\sigma^{\mu\nu} \theta)_\alpha w_{\mu\nu} - \frac{1}{2} (\sigma^\mu \bar{\theta})_\alpha \xi_\mu + \theta^2 \zeta_\alpha + \bar{\theta}^2 \epsilon_\alpha + \theta_\alpha (\eta_\mu \sigma^\mu \bar{\theta}) \\ & + \frac{1}{2} \bar{\theta}^2 \theta_\alpha \varphi - \frac{1}{2} \bar{\theta}^2 (\sigma^{\mu\nu} \theta)_\alpha \lambda_{\mu\nu} + \theta^2 (\sigma^\mu \bar{\theta})_\alpha \kappa_\mu - 2\theta^2 \bar{\theta}^2 \rho_\alpha, \\ \Omega^\mu = & \omega^\mu + \theta \zeta_\Omega^\mu + \theta^2 F_\Omega^\mu - i (\theta \sigma^\nu \bar{\theta}) \partial_\nu \omega^\mu - \frac{i}{2} \theta^2 (\bar{\theta} \bar{\sigma}^\nu \partial_\nu \zeta_\Omega^\mu) - \frac{1}{4} \theta^2 \bar{\theta}^2 \square_4 \omega^\mu, \end{aligned} \quad (3.2)$$

where $\square_4 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$, and $w_{\mu\nu}$ and $\lambda_{\mu\nu}$ are real anti-symmetric, while the others are complex. The transformation laws for the component fields of U^μ are read off from (2.15) as

$$\begin{aligned}
\delta_{\text{sc}} u^\mu &= \xi_{\text{R}}^\mu + \omega_{\text{I}}^\mu, \\
\delta_{\text{sc}} \chi_\alpha^{U\mu} &= - \left(\sigma^\mu \bar{\epsilon} - \frac{1}{2} \sigma^\mu \bar{\sigma}^\nu \eta_\nu - \frac{i}{2} \sigma^\nu \bar{\sigma}^\mu \partial_\nu l + \frac{i}{2} \zeta_\Omega^\mu \right)_\alpha, \\
\delta_{\text{sc}} a^\mu &= -\kappa^\mu + \frac{i}{2} \partial^\mu v - \frac{i}{2} \partial_\nu w^{\nu\mu} - \frac{1}{4} \epsilon^{\mu\nu\rho\tau} \partial_\nu w_{\rho\tau} - \frac{i}{2} F_\Omega^\mu, \\
\delta_{\text{sc}} \tilde{e}_\nu{}^\mu &= -\delta_\nu{}^\mu \varphi_{\text{R}} + \lambda_\nu{}^\mu + \frac{1}{2} \left(\partial^\mu \xi_{\text{I}\nu} + \partial_\nu \xi_{\text{I}}^\mu - \delta_\nu{}^\mu \partial_\rho \xi_{\text{I}}^\rho + \epsilon^\mu{}_{\nu\rho\tau} \partial^\rho \xi_{\text{R}}^\tau \right) - \partial_\nu \omega_{\text{R}}^\mu, \\
\delta_{\text{sc}} \tilde{\psi}_\alpha^\mu &= \left(2\sigma^\mu \bar{\rho} + \frac{i}{2} \sigma^\nu \bar{\sigma}^\mu \partial_\nu \epsilon - \frac{i}{2} \sigma^\nu \partial^\mu \bar{\eta}_\nu + \frac{1}{4} \sigma^\nu \partial_\nu \bar{\zeta}_\Omega^\mu \right)_\alpha, \\
\delta_{\text{sc}} d^\mu &= -\frac{1}{2} \partial^\mu \varphi_{\text{I}} + \frac{1}{4} \epsilon^{\mu\nu\rho\tau} \partial_\nu \lambda_{\rho\tau} - \frac{1}{4} \square_4 \omega_{\text{I}}^\mu,
\end{aligned} \tag{3.3}$$

where the subscript R and I denote the real and imaginary parts, respectively. By using the freedom of Ω^μ , we can set

$$u^\mu = \chi_\alpha^{U\mu} = a^\mu = 0. \tag{3.4}$$

This is analogous to the Wess-Zumino gauge for a gauge superfield. This gauge is preserved if the transformation parameters satisfy the following relations.

$$\begin{aligned}
\omega_{\text{I}}^\mu &= -\xi_{\text{R}}^\mu, \\
\zeta_{\Omega\alpha}^\mu &= (2i\sigma^\mu \bar{\epsilon} - i\sigma^\mu \bar{\sigma}^\nu \eta_\nu + \sigma^\nu \bar{\sigma}^\mu \partial_\nu l)_\alpha, \\
F_\Omega^\mu &= 2i\kappa^\mu + \partial^\mu v - \partial_\nu w^{\nu\mu} + \frac{i}{2} \epsilon^{\mu\nu\rho\tau} \partial_\nu w_{\rho\tau}.
\end{aligned} \tag{3.5}$$

We further impose an additional condition,

$$\xi_{\text{R}}^\mu = 0. \tag{3.6}$$

Then the transformation laws for the residual symmetries reduce to

$$\begin{aligned}
\delta_{\text{sc}} \tilde{e}_\nu{}^\mu &= -\delta_\nu{}^\mu \tilde{\varphi}_{\text{R}} + \tilde{\lambda}_\nu{}^\mu + \partial_\nu \tilde{\xi}_{\text{I}}^\mu, \\
\delta_{\text{sc}} \tilde{\psi}_\alpha^\mu &= (2\sigma^\mu \bar{\tilde{\rho}} + i\sigma^\nu \bar{\sigma}^\mu \partial_\nu \epsilon)_\alpha, \\
\delta_{\text{sc}} d^\mu &= -\frac{1}{2} \partial^\mu \varphi_{\text{I}} + \frac{1}{4} \epsilon^{\mu\nu\rho\tau} \partial_\nu \tilde{\lambda}_{\rho\tau},
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
\tilde{\varphi}_{\text{R}} &\equiv \varphi_{\text{R}} + \frac{1}{2} \partial_\mu \xi_{\text{I}}^\mu, \\
\tilde{\lambda}_{\mu\nu} &\equiv \lambda_{\mu\nu} + \frac{1}{2} (\partial_\mu \xi_{\text{I}\nu} - \partial_\nu \xi_{\text{I}\mu}), \\
\tilde{\xi}_{\text{I}}^\mu &\equiv \xi_{\text{I}}^\mu - \omega_{\text{R}}^\mu, \\
\tilde{\rho}^\alpha &\equiv \rho^\alpha + \frac{i}{8} (\partial_\nu \eta_\mu \sigma^\mu \bar{\sigma}^\nu)^\alpha + \frac{1}{8} \square_4 l^\alpha.
\end{aligned} \tag{3.8}$$

In fact, we can identify $\tilde{\xi}_{I\mu}$, ϵ_α , $\tilde{\lambda}_{\mu\nu}$, $\tilde{\varphi}_R$, $-\frac{2}{3}\varphi_I$ and $\tilde{\rho}_\alpha$ with the transformation parameters for \mathbf{P} , \mathbf{Q} , \mathbf{M} , \mathbf{D} , $U(1)_A$ and \mathbf{S} , respectively. They are also expressed as

$$\begin{aligned}\tilde{\xi}_I^\mu &= -\text{Re} \left(i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu \right) \Big|_0, & \epsilon^\alpha &= -\frac{1}{4} \bar{D}^2 L^\alpha \Big|_0, \\ \tilde{\lambda}_{\mu\nu} &= -\frac{1}{2} \text{Re} \left\{ (\sigma_{\mu\nu})_\beta^\alpha D_\alpha \bar{D}^2 L^\beta \right\} \Big|_0, & \tilde{\varphi}_R &= \text{Re} \left(\frac{1}{4} D^\alpha \bar{D}^2 L_\alpha \right) \Big|_0, \\ -\frac{2}{3}\varphi_I &= \text{Im} \left(-\frac{1}{6} D^\alpha \bar{D}^2 L_\alpha \right) \Big|_0, & \tilde{\rho}^\alpha &= -\frac{1}{32} D^2 \bar{D}^2 L^\alpha \Big|_0,\end{aligned}\tag{3.9}$$

where the symbol $|_0$ denotes the lowest component of a superfield.

With the above identification of the parameters, the transformations in (3.7) agree with those for the Weyl multiplet in Ref. [6], which corresponds to the SUGRA multiplet, if we specify the components of U^μ as

$$\begin{aligned}\tilde{e}_\nu{}^\mu &= e_\nu{}^\mu - \delta_\nu{}^\mu, \\ \tilde{\psi}_\alpha^\mu &= i(\sigma^\nu \bar{\sigma}^\mu \psi_\nu)_\alpha, \\ d^\mu &= \frac{3}{4} A^\mu - \frac{1}{4} \epsilon^{\mu\nu\rho\tau} \partial_\nu \tilde{e}_{\rho\tau},\end{aligned}\tag{3.10}$$

where $e_\nu{}^\mu$, ψ_μ and A_μ are the vierbein, the gravitino and the $U(1)_A$ gauge field.⁴ Namely, $\tilde{e}_\nu{}^\mu$ is the fluctuation of the vierbein since $\langle e_\nu{}^\mu \rangle = \delta_\nu{}^\mu$, and the (linearized) transformation laws of ψ_μ and A_μ are given by [6]

$$\delta_{\text{sc}} \psi_{\mu\alpha} = \partial_\mu \epsilon_\alpha + i(\sigma_\mu \bar{\tilde{\rho}})_\alpha, \quad \delta_{\text{sc}} A_\mu = -\frac{2}{3} \partial_\mu \varphi_I.\tag{3.11}$$

In the subsequent subsections, we compare the transformation laws for component fields in each superfield with those in the superconformal formulation [4], and identify the component fields. The transformation laws in Ref. [4] are compactly summarized in Sec. 3 of Ref. [6]. Hence we basically use the notations of Ref. [6] as the component fields in each multiplet.

3.2 Chiral multiplet

Now we consider the transformation laws of a chiral superfield Φ . In this subsection, we work in the chiral coordinate $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar{\theta}$. (Recall our definition of $\bar{D}_{\dot{\alpha}}$ in (2.1).) Then, it is expanded as

$$\Phi = \phi + \theta\chi + \theta^2 F.\tag{3.12}$$

⁴ The $U(1)_A$ gauge field A_μ is an auxiliary field [4].

Focusing on terms proportional to the Weyl weight in the transformation laws in Ref. [6], the chiral superfield Λ in (2.13) is identified. We find that Λ cannot be expressed by only L^α and Ω^μ . A choice of $\Lambda = -\frac{1}{24}\bar{D}^2 D^\alpha L_\alpha$ reproduces the correct $U(1)_A$ transformation, but there are extra terms for other superconformal transformations. Fortunately such extra terms are summarized in the form of a chiral superfield. Thus there exists a choice of Λ that realizes the correct superconformal transformations for a chiral multiplet. It is given by

$$\Lambda = -\frac{1}{24} (\bar{D}^2 D^\alpha L_\alpha + 4\Xi), \quad (3.13)$$

where

$$\Xi \equiv (-4\tilde{\varphi}_R + \partial_\mu \xi_I^\mu) + 8\theta \left(\tilde{\rho} - \frac{i}{8} \sigma^\nu \bar{\sigma}^\mu \partial_\nu \eta_\mu + \frac{1}{8} \square_4 l \right) + 2i\theta^2 \partial_\mu \left(\kappa^\mu - \frac{i}{2} \partial^\mu v \right). \quad (3.14)$$

Then, the transformation laws of the component fields are read off as

$$\begin{aligned} \delta_{\text{sc}} \phi &= \tilde{\xi}_I^\mu \partial_\mu \phi + \epsilon \chi + w \tilde{\varphi}_R \phi - \frac{iw}{3} \varphi_I \phi, \\ \delta_{\text{sc}} \chi_\alpha &= \tilde{\xi}_I^\mu \partial_\mu \chi_\alpha + \frac{1}{2} \tilde{\lambda}_{\mu\nu} (\sigma^{\mu\nu} \chi)_\alpha + 2\epsilon_\alpha F - 2i (\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi \\ &\quad + \left(w + \frac{1}{2} \right) \tilde{\varphi}_R \chi_\alpha - \frac{i}{3} \left(w - \frac{3}{2} \right) \varphi_I \chi_\alpha - 4w \tilde{\rho}_\alpha \phi, \\ \delta_{\text{sc}} F &= \tilde{\xi}_I^\mu \partial_\mu F - i\bar{\epsilon} \bar{\sigma}^\mu \partial_\mu \chi + (w+1) \tilde{\varphi}_R F - \frac{i}{3} (w-3) \varphi_I F + 2(w-1) \tilde{\rho} \chi. \end{aligned} \quad (3.15)$$

These transformations agree with those in Ref. [6].

Let us comment on a chiral superfield Φ in the full superspace integral $\int d^4\theta$. Unlike the global SUSY case, moving the bases from the chiral coordinate y^μ to the original one x^μ is not enough. In fact, Φ must appear in the form of

$$\mathcal{V}(\Phi) = (1 + iU^\mu \partial_\mu) \Phi. \quad (3.16)$$

This is regarded as the embedding of the chiral multiplet into a general multiplet as mentioned below (2.18). The embedded superfield has the following components.

$$\begin{aligned} \mathcal{V}(\Phi) &= \phi + \theta \chi + \theta^2 F - i (\theta \sigma^\mu \bar{\theta}) (e^{-1})_\mu^\nu \partial_\nu \phi - \frac{i}{2} \theta^2 \left\{ \bar{\theta} \bar{\sigma}^\mu (e^{-1})_\mu^\nu \partial_\nu \chi - 2 (\bar{\theta} \tilde{\psi}^\mu) \partial_\mu \phi \right\} \\ &\quad + i \bar{\theta}^2 (\bar{\theta} \tilde{\psi}^\mu) \partial_\mu \phi - \frac{1}{4} \theta^2 \bar{\theta}^2 \left\{ g^{\mu\nu} \partial_\mu \partial_\nu \phi + 2i \tilde{\psi}^\mu \partial_\mu \chi - 4i d^\mu \partial_\mu \phi \right\}, \end{aligned} \quad (3.17)$$

where $(e^{-1})_\mu^\nu \equiv \delta_\mu^\nu - \tilde{e}_\mu^\nu$ and $g^{\mu\nu} \equiv \eta^{\mu\nu} - \tilde{e}^{\mu\nu} - \tilde{e}^{\nu\mu}$ are the inverse matrices of the vierbein and the metric, respectively.

3.3 Real general multiplet

Next we consider a real general superfield V . From (2.14) with (3.13), its transformation law is given by

$$\delta_{\text{sc}} V = \left\{ -\frac{1}{4} \bar{D}^2 L^\alpha D_\alpha - \frac{1}{2} (i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu - \frac{w}{24} (\bar{D}^2 D^\alpha L_\alpha + 4\Xi) + \text{h.c.} \right\} V. \quad (3.18)$$

Each component of V is defined as

$$V = C' + i\theta\zeta' - i\bar{\theta}\bar{\zeta}' - \theta^2\mathcal{H}' - \bar{\theta}^2\bar{\mathcal{H}}' - (\theta\sigma^\mu\bar{\theta}) B'_\mu + i\theta^2(\bar{\theta}\bar{\lambda}') - i\bar{\theta}^2(\theta\lambda') + \frac{1}{2}\theta^2\bar{\theta}^2 D', \quad (3.19)$$

where C' , B'_μ and D' are real. By expanding (3.18) in terms of the components, we obtain their transformation laws. They do not agree with the transformation laws in Ref. [6] as is. To reproduce the latter laws, we need to redefine the components as

$$\begin{aligned} C &\equiv C', & \zeta_\alpha &\equiv \zeta'_\alpha, & \mathcal{H} &\equiv \mathcal{H}', \\ B_\mu &\equiv B'_\mu + \zeta'_\nu \psi_\mu + \bar{\zeta}'_\nu \bar{\psi}_\mu + \frac{w}{2} C' A_\mu, \\ \lambda_\alpha &\equiv \lambda'_\alpha + \frac{i}{2} \left\{ \sigma^\mu (e^{-1})_\mu{}^\nu \partial_\nu \bar{\zeta}' \right\}_\alpha + (\sigma^\mu \bar{\sigma}^\nu \psi_\mu)_\alpha B'_\nu + \frac{w}{4} (\sigma^\mu \bar{\zeta}')_\alpha A_\mu, \\ D &\equiv D' + \frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu C' + \left(\bar{\lambda}' \bar{\sigma}^\mu \psi_\mu - \frac{i}{2} \partial_\nu \zeta' \sigma^\mu \bar{\sigma}^\nu \psi_\mu - i \partial_\mu \zeta' \psi^\mu - \frac{2iw}{3} \zeta' \sigma^{\mu\nu} \partial_\nu \psi_\mu + \text{h.c.} \right) \\ &\quad - \left(2d^\mu - \frac{w}{2} A^\mu \right) B'_\mu. \end{aligned} \quad (3.20)$$

The explicit form of V in terms of these redefined components is shown in (A.3). Then we obtain

$$\begin{aligned} \delta_{\text{sc}} C &= \tilde{\xi}_I^\mu \partial_\mu C + i\epsilon\zeta - i\bar{\epsilon}\bar{\zeta} + w\tilde{\varphi}_R C, \\ \delta_{\text{sc}} \zeta_\alpha &= \tilde{\xi}_I^\mu \partial_\mu \zeta_\alpha + \frac{1}{2} \tilde{\lambda}_{\mu\nu} (\sigma^{\mu\nu} \zeta)_\alpha + 2i\epsilon_\alpha \mathcal{H} + (\sigma^\mu \bar{\epsilon})_\alpha (iB_\mu - \partial_\mu C) \\ &\quad + \left(w + \frac{1}{2} \right) \tilde{\varphi}_R \zeta_\alpha + \frac{i}{2} \varphi_I \zeta_\alpha + 2iw\tilde{\rho}_\alpha C, \\ \delta_{\text{sc}} \mathcal{H} &= \tilde{\xi}_I^\mu \partial_\mu \mathcal{H} - i\bar{\epsilon}\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \zeta + (w+1) \tilde{\varphi}_R \mathcal{H} + i\varphi_I \mathcal{H} - i(w-2) \tilde{\rho} \zeta, \\ \delta_{\text{sc}} B_\mu &= \tilde{\xi}_I^\nu \partial_\nu B_\mu + \tilde{\lambda}_{\mu\nu} B^\nu - i\epsilon\sigma_\mu \bar{\lambda} - i\bar{\epsilon}\bar{\sigma}_\mu \lambda - \epsilon\partial_\mu \zeta - \bar{\epsilon}\partial_\mu \bar{\zeta} \\ &\quad + (w+1) \tilde{\varphi}_R B_\mu - i(w+1) (\tilde{\rho}\sigma_\mu \bar{\zeta} + \bar{\rho}\bar{\sigma}_\mu \zeta), \\ \delta_{\text{sc}} \lambda_\alpha &= \tilde{\xi}_I^\mu \partial_\mu \lambda_\alpha + \frac{1}{2} \tilde{\lambda}_{\mu\nu} (\sigma^{\mu\nu} \lambda)_\alpha + i\epsilon_\alpha D - (\sigma^{\mu\nu} \epsilon)_\alpha (\partial_\mu B_\nu - \partial_\nu B_\mu) + \left(w + \frac{3}{2} \right) \tilde{\varphi}_R \lambda_\alpha \\ &\quad + \frac{iw}{2} \partial_\mu \tilde{\varphi}_R (\sigma^\mu \bar{\zeta})_\alpha - \frac{i}{2} \varphi_I \lambda_\alpha - iw(\sigma^\mu \bar{\rho})_\alpha B_\mu + 2iw\tilde{\rho}_\alpha \mathcal{H} + w(\sigma^\mu \bar{\rho})_\alpha \partial_\mu C, \\ \delta_{\text{sc}} D &= \tilde{\xi}_I^\mu \partial_\mu D + 2\partial_\mu \tilde{\lambda}^{\mu\nu} \partial_\nu C + (w+2) \tilde{\varphi}_R D + w\partial_\mu \tilde{\varphi}_R \partial^\mu C \\ &\quad - \{ \bar{\epsilon}\bar{\sigma}^\mu \partial_\mu \lambda + 2iw\tilde{\rho}\lambda + w\tilde{\rho}\sigma^\mu \partial_\mu \bar{\zeta} + \text{h.c.} \}. \end{aligned} \quad (3.21)$$

These transformation laws agree with those in Ref. [6] at the linearized level,⁵ except for the following two points.

- The second term in $\delta_{\text{sc}} D$ is absent in Ref. [6]. However, this term is harmless when we consider the invariance of the action because D is the highest component and this term is a total derivative.
- The terms proportional to $\partial_\mu \tilde{\varphi}_R$ in $\delta_{\text{sc}} \lambda_\alpha$ and $\delta_{\text{sc}} D$ also seem to be extra terms that are absent in Ref. [6], at first glance. Actually, they indicate that the \mathbf{D} -gauge field b_μ is set to zero in our linearized SUGRA. Its superconformal transformation (at the linearized level) is

$$\delta_{\text{sc}} b_\mu = \partial_\mu \tilde{\varphi}_R - 2\xi_{K\mu}, \quad (3.22)$$

where $\xi_{K\mu}$ is the transformation parameter for \mathbf{K} . Thus, keeping the condition $b_\mu = 0$ requires $\xi_{K\mu} = \frac{1}{2} \partial_\mu \tilde{\varphi}_R$. In fact, after the replacement: $\partial_\mu \tilde{\varphi}_R \rightarrow 2\xi_{K\mu}$, (3.21) reproduces the correct transformations.

3.4 Gauge multiplet

Here we consider a gauge multiplet, which corresponds to a real general multiplet with $w = 0$.⁶ From (3.20), such a real general superfield V is expressed as

$$\begin{aligned} V = & C + i\theta\zeta - i\bar{\theta}\bar{\zeta} - \theta^2\mathcal{H} - \bar{\theta}^2\bar{\mathcal{H}} - (\theta\sigma^\mu\bar{\theta}) (e^{-1})_\mu{}^\nu \hat{B}_\nu \\ & + i\theta^2\bar{\theta} \left\{ \bar{\lambda} - \frac{i}{2}\bar{\sigma}^\mu (e^{-1})_\mu{}^\nu \partial_\nu \zeta - i\bar{\psi}_\mu \hat{B}^\mu \right\} - i\bar{\theta}^2\theta \left\{ \lambda - \frac{i}{2}\sigma^\mu (e^{-1})_\mu{}^\nu \partial_\nu \bar{\zeta} + i\tilde{\psi}_\mu \hat{B}^\mu \right\} \\ & + \frac{1}{2}\theta^2\bar{\theta}^2 \left\{ D - \frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu C + \left(-\frac{i}{2}\bar{\lambda}\bar{\sigma}^\mu\tilde{\psi}_\mu + \partial_\mu\zeta\tilde{\psi}^\mu + \text{h.c.} \right) + 2d^\mu \hat{B}_\mu \right\}, \end{aligned} \quad (3.23)$$

where

$$\hat{B}_\mu \equiv (\delta_\mu{}^\nu + \tilde{e}_\mu{}^\nu) B_\nu - \zeta\psi_\mu - \bar{\zeta}\bar{\psi}_\mu, \quad (3.24)$$

is interpreted as a gauge field. This definition of the gauge field is consistent with that of Ref. [4].

The gauge transformation can be defined just in a similar way to the global SUSY case as

$$V \rightarrow V + \mathcal{V}(\Sigma) + \mathcal{V}(\bar{\Sigma}), \quad (3.25)$$

⁵ The complex scalar \mathcal{H} should be understood as $\frac{1}{2}(H + iK)$ in the notation of Ref. [6].

⁶ We consider an abelian gauge multiplet for simplicity. An extension to the nonabelian case is straightforward.

where $\Sigma = \phi_\Sigma + \theta\chi_\Sigma + \theta^2 F_\Sigma$ (in the coordinate $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$) is a chiral superfield. Note that Σ must be embedded into a general multiplet by \mathcal{V} in order to be added to V . We can move to the Wess-Zumino gauge by choosing Σ as

$$\text{Re } \phi_\Sigma = -\frac{1}{2}C, \quad \chi_{\Sigma\alpha} = -i\zeta_\alpha, \quad F_\Sigma = \mathcal{H}. \quad (3.26)$$

In this gauge, V is written as

$$\begin{aligned} V_{\text{WZ}} = & -(\theta\sigma^\mu\bar{\theta}) (e^{-1})_\mu{}^\nu \hat{B}'_\nu + i\theta^2\bar{\theta} \left(\bar{\lambda} - i\tilde{\psi}^\mu \hat{B}'_\mu \right) - i\bar{\theta}^2\theta \left(\lambda + i\tilde{\psi}^\mu \hat{B}'_\mu \right) \\ & + \frac{1}{2}\theta^2\bar{\theta}^2 \left\{ D + \left(-\frac{i}{2}\bar{\lambda}\bar{\sigma}^\mu\tilde{\psi}_\mu + \text{h.c.} \right) + 2d^\mu \hat{B}'_\mu \right\}, \end{aligned} \quad (3.27)$$

where $\hat{B}'_\mu \equiv \hat{B}_\mu - 2\partial_\mu \text{Im } \phi_\Sigma$ is the gauge-transformed gauge field. We can move to this gauge only when $w = 0$. The set of the components $[\hat{B}'_\mu, \lambda_\alpha, D]$ form a gauge multiplet in Ref. [4, 6].

Next we construct a field strength superfield \mathcal{W}_α that is gauge-invariant from the gauge superfield V . A naive definition of \mathcal{W}_α ,

$$\mathcal{W}_\alpha^{\text{naive}} \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V, \quad (3.28)$$

is not invariant under (3.25). If we define

$$X \equiv \left(1 + \frac{1}{4}U^\mu \bar{\sigma}_\mu^{\dot{\beta}\beta} [D_\beta, \bar{D}_{\dot{\beta}}] \right) V, \quad (3.29)$$

its gauge transformation becomes simpler,

$$X \rightarrow X + \Sigma + \bar{\Sigma}. \quad (3.30)$$

Hence, $-\frac{1}{4}\bar{D}^2 D_\alpha X$ becomes gauge-invariant. However, this is not the only way to construct a gauge-invariant quantity. We can define the following quantity by adding the second term that is also gauge-invariant.

$$\hat{\mathcal{W}}_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha X + c\bar{D}^2 \left(U^\mu \sigma_\mu^{\dot{\beta}\beta} D_\alpha D_\beta \bar{D}_{\dot{\beta}} V \right), \quad (3.31)$$

where c is a constant to be determined later. The second term in (3.31) does not contribute to the lowest component, and we find that

$$\hat{\mathcal{W}}_\alpha = -i \left(\lambda + \frac{1}{2}\tilde{e}_\mu{}^\nu \sigma^\mu \bar{\sigma}_\nu \lambda \right)_\alpha + \mathcal{O}(\theta). \quad (3.32)$$

This indicates that we have to multiply $\hat{\mathcal{W}}_\alpha$ by a superfield $Z_\alpha^\beta = \delta_\alpha^\beta - \frac{1}{2}\tilde{e}_\mu{}^\nu (\sigma^\mu \bar{\sigma}_\nu)_\alpha{}^\beta + \mathcal{O}(\theta)$ in order to obtain the desired field strength superfield whose lowest component is $-i\lambda_\alpha$. The higher components of Z_α^β and the constant c in (3.31) are determined so that $\mathcal{W}_\alpha \equiv Z_\alpha^\beta \hat{\mathcal{W}}_\beta$ has the correct components. The result is

$$\begin{aligned}\mathcal{W}_\alpha &= -\frac{1}{4}Z_\alpha^\beta \bar{D}^2 \left\{ D_\beta \left(V + \frac{1}{4}U^\mu \bar{\sigma}_\mu^{\dot{\gamma}\gamma} [D_\gamma, \bar{D}_{\dot{\gamma}}] V \right) - \frac{1}{2}U^\mu \bar{\sigma}_\mu^{\dot{\gamma}\gamma} D_\beta D_\gamma \bar{D}_{\dot{\gamma}} V \right\} \\ &= -\frac{1}{4}Z_\alpha^\beta \bar{D}^2 \left\{ D_\beta V + \frac{1}{4}D_\beta U^\mu \bar{\sigma}_\mu^{\dot{\gamma}\gamma} [D_\gamma, \bar{D}_{\dot{\gamma}}] V - iU^\mu \partial_\mu D_\beta V \right\}, \\ Z_\alpha^\beta &\equiv \delta_\alpha^\beta - \frac{1}{2}\tilde{e}_\mu{}^\nu (\sigma^\mu \bar{\sigma}_\nu)_\alpha{}^\beta - \left(\sigma^\mu \tilde{\psi}_\mu \right)_\alpha \theta^\beta,\end{aligned}\tag{3.33}$$

where Z_α^β is expressed in the chiral coordinate y^μ . Each component of \mathcal{W}_α is calculated as

$$\mathcal{W}_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha (e^{-1})_\mu{}^\rho (e^{-1})_\nu{}^\tau \hat{F}_{\rho\tau} - \theta^2 \left\{ \sigma^\mu (e^{-1})_\mu{}^\nu \mathcal{D}_\nu \bar{\lambda} \right\}_\alpha, \tag{3.34}$$

where

$$\begin{aligned}\hat{F}_{\mu\nu} &\equiv \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + i(\psi_\mu \sigma_\nu \bar{\lambda} - \psi_\nu \sigma_\mu \bar{\lambda}) + i(\bar{\psi}_\mu \bar{\sigma}_\nu \lambda - \bar{\psi}_\nu \bar{\sigma}_\mu \lambda), \\ (\mathcal{D}_\mu \bar{\lambda})^{\dot{\alpha}} &\equiv \left\{ \left(\partial_\mu - \frac{1}{2}\omega_\mu{}^{\nu\rho} \sigma_{\nu\rho} + \frac{3i}{4}A_\mu \right) \bar{\lambda} \right\}^{\dot{\alpha}} + (\bar{\sigma}^{\nu\rho} \bar{\psi}_\mu)^{\dot{\alpha}} \hat{F}_{\nu\rho} + i\bar{\psi}_\mu^{\dot{\alpha}} D.\end{aligned}\tag{3.35}$$

Here $\omega_\mu{}^{\nu\rho}$ is the spin connection, and expressed at the linearized level as

$$\omega_\mu{}^{\nu\rho} = -\frac{1}{2}\partial_\mu (\tilde{e}^{\nu\rho} - \tilde{e}^{\rho\nu}) + \frac{1}{2} \left\{ \partial^\nu (\tilde{e}_\mu{}^\rho + \tilde{e}^\rho{}_\mu) - \partial^\rho (\tilde{e}_\mu{}^\nu + \tilde{e}^\nu{}_\mu) \right\}. \tag{3.36}$$

The field strength $\hat{F}_{\mu\nu}$ and the covariant derivative $\mathcal{D}_\mu \bar{\lambda}$ defined in (3.35) agree with those in Ref. [6]. We have used the identification (3.10).

4 Action formulae

4.1 F -term action formula

First we consider the F -term invariant action, which consists of only chiral multiplets. It reduces to the chiral superspace integral in the global SUSY limit,

$$S_F^{\text{gl}}[W] \equiv \int d^4x \int d^2\theta W + \text{h.c.}, \tag{4.1}$$

where W is a chiral superfield, which is referred to as the superpotential. From (2.13) with (3.13), this transforms as

$$\begin{aligned}
\delta_{\text{sc}} S_F^{\text{gl}}[W] &= \int d^4x \int d^2\theta \left\{ -\frac{1}{4} \bar{D}^2 (L^\alpha D_\alpha W) - \Omega^\mu \partial_\mu W - \frac{1}{4} (\bar{D}^2 D^\alpha L_\alpha + 4\Xi) W \right\} + \text{h.c.} \\
&= \int d^4x \int d^2\theta \left\{ -\frac{1}{4} \bar{D}^2 D^\alpha (L_\alpha W) - \Omega^\mu \partial_\mu W - \Xi W \right\} + \text{h.c.} \\
&= \int d^4x \left[\int d^4\theta D^\alpha (L_\alpha W) - \int d^2\theta (\Xi - \partial_\mu \Omega^\mu) W \right] + \text{h.c.} \tag{4.2}
\end{aligned}$$

We have assumed that $w = n = 3$ for W . In the last equation, we performed the partial integrals. In order to make the action invariant, we introduce a chiral superfield $\tilde{\mathcal{E}}$ whose transformation law is given by

$$\begin{aligned}
\delta_{\text{sc}} \tilde{\mathcal{E}} &= \Xi - \partial_\mu \Omega^\mu \\
&= \left(-4\tilde{\varphi}_{\text{R}} + \partial_\mu \tilde{\xi}_{\text{I}}^\mu \right) + 8\theta \left(\tilde{\rho} - \frac{i}{4} \sigma^\mu \partial_\mu \bar{\epsilon} \right), \tag{4.3}
\end{aligned}$$

in the coordinate y^μ , and modify the action formula as

$$S_F[W] \equiv \int d^4x \int d^2\theta \left(1 + \tilde{\mathcal{E}} \right) W + \text{h.c.} \tag{4.4}$$

From (3.7) and (4.3), we identify $\tilde{\mathcal{E}}$ as

$$\tilde{\mathcal{E}} = \tilde{e}_\mu{}^\mu - \theta \sigma^\mu \tilde{\psi}_\mu = \tilde{e}_\mu{}^\mu - 2i\theta \sigma^\mu \bar{\psi}_\mu. \tag{4.5}$$

Therefore, the factor $(1 + \tilde{\mathcal{E}})$ in (4.4) corresponds to the chiral density multiplet in Ref. [11].⁷

The action $S_F[W]$ is now invariant under δ_{sc} at the linearized level. Note that it is invariant only when the Weyl weight of W is 3. This is consistent with the F -term action formula in Ref. [4], which is shown in (B.1) in our notations. We can explicitly see that (4.4) reproduces (B.1).

4.2 D -term action formula

Next we consider the D -term invariant action. It reduces to the full superspace integral in the global SUSY limit,⁸

$$S_D^{\text{gl}}[K] \equiv 2 \int d^4x \int d^4\theta K, \tag{4.6}$$

⁷ Note that $\det(e_\mu{}^\mu) = 1 + \tilde{e}_\mu{}^\mu$ at the linearized level.

⁸ The factor 2 is necessary to match the normalization of the D -term action formula in Ref. [4].

where K is a real general superfield, which is referred to as the Kähler potential. From (2.14) with (3.13), this transforms as

$$\begin{aligned}\delta_{\text{sc}} S_D^{\text{gl}}[K] &= 2 \int d^4x \int d^4\theta \left\{ -\frac{1}{4} \bar{D}^2 L^\alpha D_\alpha - \frac{1}{2} (i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu \right. \\ &\quad \left. - \frac{w}{24} (\bar{D}^2 D^\alpha L_\alpha + 4\Xi) + \text{h.c.} \right\} K \\ &= 2 \int d^4x \int d^4\theta \left\{ \frac{6-w}{24} \bar{D}^2 D^\alpha L_\alpha - \frac{1}{2} (i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{D}^{\dot{\alpha}} L^\alpha - \partial_\mu \Omega^\mu) - \frac{w}{6} \Xi + \text{h.c.} \right\} K,\end{aligned}\tag{4.7}$$

where w is the Weyl weight of K . We have performed the partial integral in the second equality. Here we define a real scalar superfield \tilde{E}_1 from the connection superfield U^μ as

$$\tilde{E}_1 \equiv \frac{1}{4} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] U^\mu.\tag{4.8}$$

Then it transforms as

$$\delta_{\text{sc}} \tilde{E}_1 = -\frac{1}{2} \bar{D}^2 D^\alpha L_\alpha + \frac{3i}{2} \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{D}^{\dot{\alpha}} L^\alpha - \frac{1}{2} \partial_\mu \Omega^\mu + \text{h.c.}.\tag{4.9}$$

By using \tilde{E}_1 and $\tilde{\mathcal{E}}$ defined in (4.5), we modify the action formula as

$$S_D[K] \equiv 2 \int d^4x \int d^4\theta \left\{ 1 + \frac{1}{3} (\tilde{E}_1 + \tilde{\mathcal{E}} + \bar{\tilde{\mathcal{E}}}) \right\} K,\tag{4.10}$$

so that its transformation becomes

$$\delta_{\text{sc}} S_D[K] = 2 \int d^4x \int d^4\theta \left\{ \frac{2-w}{24} \bar{D}^2 D^\alpha L_\alpha + \frac{2-w}{6} \Xi + \text{h.c.} \right\} K.\tag{4.11}$$

Therefore, $S_D[K]$ is now δ_{sc} -invariant when $w = 2$, and can be identified with the D -term action formula in Ref. [4], which is shown in (B.2) in our notations. Since the prefactor of K in (4.10) is expanded as (see (A.5))

$$1 + \frac{1}{3} (\tilde{E}_1 + \tilde{\mathcal{E}} + \bar{\tilde{\mathcal{E}}}) = 1 + \tilde{e}_\mu{}^\mu + \mathcal{O}(\theta^2),\tag{4.12}$$

this corresponds to the density multiplet in the full superspace [11]. We can explicitly show that (4.10) reproduces (B.2), except for the kinetic terms for the SUGRA fields, which will be discussed in Sec. 4.4.

4.3 Absorption of chiral density superfield

Notice that the ‘‘chiral density superfield’’ $\tilde{\mathcal{E}}$ defined in (4.5) is redundant because it cannot be expressed in terms of U^μ and the SUGRA fields are already contained in the latter. In

fact, we can eliminate $\tilde{\mathcal{E}}$ from the action formulae (4.4) and (4.10) by the following superfield redefinition.

$$\begin{aligned}\hat{\Phi} &\equiv \left(1 + \frac{w}{3}\tilde{\mathcal{E}}\right)\Phi, \\ \hat{V} &\equiv \left\{1 + \frac{w}{6}\left(\tilde{\mathcal{E}} + \bar{\tilde{\mathcal{E}}}\right)\right\}V,\end{aligned}\tag{4.13}$$

where Φ and V are a chiral and a real general superfields, respectively. The redefined superfields transform as

$$\begin{aligned}\delta_{\text{sc}}\hat{\Phi} &= \left\{-\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - (i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu - \frac{w}{12}(\bar{D}^2 D^\alpha L_\alpha + 4\partial_\mu \Omega^\mu)\right\}\hat{\Phi}, \\ \delta_{\text{sc}}\hat{V} &= \left\{-\frac{1}{4}\bar{D}^2 L^\alpha D_\alpha - \frac{1}{2}(i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu) \partial_\mu - \frac{w}{24}(\bar{D}^2 D^\alpha L_\alpha + 4\partial_\mu \Omega^\mu) + \text{h.c.}\right\}\hat{V}.\end{aligned}\tag{4.14}$$

Now the transformations are expressed only in terms of L^α and Ω^μ . In terms of these redefined superfields, the action formulae are expressed as

$$\begin{aligned}S_F[W] &= \int d^4x \int d^2\theta \hat{W} + \text{h.c.}, \\ S_D[K] &= 2 \int d^4x \int d^4\theta \left(1 + \frac{1}{3}\tilde{E}_1\right) \hat{K} \\ &= 2 \int d^4x \int d^4\theta \left(1 + \frac{1}{12}\bar{\sigma}_\mu^{\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] U^\mu\right) \hat{K}.\end{aligned}\tag{4.15}$$

As for the gauge kinetic term, the $\tilde{\mathcal{E}}$ -dependence automatically cancels with another redundant superfield $Z_\alpha{}^\beta$ in (3.33) because

$$\begin{aligned}\mathcal{W}_\alpha &= Z_\alpha{}^\beta \hat{\mathcal{W}}_\beta = \hat{\mathcal{W}}_\alpha - \frac{1}{2}\tilde{e}_\mu{}^\nu \left(\sigma^\mu \bar{\sigma}_\nu \hat{\mathcal{W}}\right)_\alpha - \left(\theta \hat{\mathcal{W}}\right) \left(\sigma^\mu \bar{\tilde{\psi}}_\mu\right)_\alpha, \\ \mathcal{W}^\alpha \mathcal{W}_\alpha &= \left\{1 - \tilde{e}_\mu{}^\mu + \theta \sigma^\mu \bar{\tilde{\psi}}_\mu\right\} \hat{\mathcal{W}}^\alpha \hat{\mathcal{W}}_\alpha = \left(1 - \tilde{\mathcal{E}}\right) \hat{\mathcal{W}}^\alpha \hat{\mathcal{W}}_\alpha.\end{aligned}\tag{4.16}$$

Thus we obtain

$$\begin{aligned}S_{\text{kin}}^{\text{gauge}}[V] &\equiv S_F \left[-\frac{1}{4}\mathcal{W}^\alpha \mathcal{W}_\alpha\right] \\ &= \int d^4x \int d^2\theta \left(1 + \tilde{\mathcal{E}}\right) \left(-\frac{1}{4}\mathcal{W}^\alpha \mathcal{W}_\alpha\right) + \text{h.c.} \\ &= \int d^4x \int d^2\theta \left(-\frac{1}{4}\hat{\mathcal{W}}^\alpha \hat{\mathcal{W}}_\alpha\right) + \text{h.c.},\end{aligned}\tag{4.17}$$

where

$$\hat{\mathcal{W}}_\alpha = -\frac{1}{4}\bar{D}^2 \left(D_\alpha \hat{V} + \frac{1}{4}D_\alpha U^\mu \bar{\sigma}_\mu^{\dot{\beta}\beta} [D_\beta, \bar{D}_{\dot{\beta}}] \hat{V} - iU^\mu \partial_\mu D_\alpha \hat{V}\right).\tag{4.18}$$

Note that $\hat{V} = V$ since the Weyl weight of the gauge superfield is zero.

4.4 Kinetic terms for SUGRA fields

Here we discuss the kinetic terms for the SUGRA superfield U^μ . In the superconformal formulation, the corresponding terms are contained in the D -term action formula in Ref. [4] as follows. (See eq.(B.2).)

$$S_D[\Omega] = e \left[D + \cdots + \frac{C}{3} \{ R(\omega) + 4\epsilon^{\mu\nu\rho\tau} (\psi_\mu \sigma_\tau \partial_\nu \bar{\psi}_\rho + \text{h.c.}) \} \right], \quad (4.19)$$

where e is the determinant of the vierbein, $R(\omega)$ is the scalar curvature constructed from the spin connection, and $\Omega = [C, \cdots, D]$ is a real general multiplet with the Weyl weight 2. The Einstein-Hilbert term is obtained by imposing the \mathbf{D} -gauge fixing condition, $C = -\frac{3}{2}$.⁹ Since the above kinetic terms are quadratic in the SUGRA fields, it seems that we need to extend the D -term action formula in (4.15) as

$$S_D[\Omega] = 2 \int d^4x \int d^4\theta \left\{ 1 + \frac{1}{3} (\tilde{E}_1 + \tilde{E}_2) \right\} \hat{\Omega}, \quad (4.20)$$

where \tilde{E}_2 is quadratic in U^μ . The quadratic part \tilde{E}_2 is specified by requiring the invariance of the action up to linear order in the SUGRA fields. However, information on higher order corrections to (4.9) and (4.14) in the SUGRA fields is necessary for this procedure, which is beyond the linearized SUGRA. Fortunately, it is possible to extend (4.15) to include the kinetic terms for U^μ without information on the higher order corrections, as we will explain below.

Recall that the SUGRA kinetic terms are proportional to the lowest component of $\hat{\Omega}$, which will be set to a constant $\Omega_0 = -\frac{3}{2}$ after the \mathbf{D} -gauge fixing. Thus we expand $\hat{\Omega}$ as

$$\hat{\Omega} = \Omega_0 + \tilde{\tilde{\Omega}}, \quad (4.21)$$

and focus on $\tilde{E}_2\Omega_0$ out of $\tilde{E}_2\hat{\Omega}$ in (4.20). Namely, we consider

$$\begin{aligned} S_D[\Omega] &= 2 \int d^4x \int d^4\theta \left\{ \left(1 + \frac{1}{3} \tilde{E}_1 \right) (\Omega_0 + \tilde{\tilde{\Omega}}) + \frac{1}{3} \tilde{E}_2 \Omega_0 \right\} \\ &= 2 \int d^4x \int d^4\theta \left\{ \frac{1}{3} \tilde{E}_2 \Omega_0 + \left(1 + \frac{1}{3} \tilde{E}_1 \right) \tilde{\tilde{\Omega}} \right\}, \end{aligned} \quad (4.22)$$

as an extension of (4.15). We have used a fact that \tilde{E}_1 is a total derivative at the second equality. For example, when $\hat{\Omega}$ is given by

$$\hat{\Omega} = -\frac{3}{2} |\hat{\Phi}_C|^2 e^{-\hat{K}/3}, \quad (4.23)$$

⁹ We have taken the unit of the Planck mass, $M_{\text{Pl}} = 1$.

where \hat{K} is quadratic in matter fields and $\hat{\Phi}_C = 1 + \dots$ is a chiral compensator superfield, the action (4.22) is written as

$$S_D[\Omega] = \int d^4x \int d^4\theta \left\{ -\tilde{E}_2 + \left(1 + \frac{1}{3}\tilde{E}_1\right) (\hat{K} + \dots) \right\}, \quad (4.24)$$

where the ellipsis denotes higher order terms.

We require the action (4.22) to be invariant up to linear order in the SUGRA fields for Ω_0 -dependent terms while up to zeroth order in the SUGRA fields for $\tilde{\Omega}$ -dependent terms. Up to this order,

$$\int d^4x \int d^4\theta \left\{ \frac{1}{3} (\delta_{\text{sc}} \tilde{E}_1) \tilde{\Omega} + \delta_{\text{sc}} \tilde{\Omega} \right\} = 0, \quad (4.25)$$

as shown in Sec. 4.2. Hence the variation of (4.22) becomes

$$\delta_{\text{sc}} S_D[\Omega] = 2 \int d^4x \int d^4\theta \left\{ \frac{1}{3} (\delta_{\text{sc}} \tilde{E}_2) \Omega_0 + \frac{1}{3} \tilde{E}_1 \delta_{\text{sc}} \tilde{\Omega} \right\}. \quad (4.26)$$

In order to discuss the invariance of the action up to the order under consideration, we only need a field-independent part of $\delta_{\text{sc}} \tilde{\Omega}$. From (4.14), it is read off as

$$\delta_{\text{sc}} \tilde{\Omega} = \delta_{\text{sc}} \hat{\Omega} = -\frac{\Omega_0}{12} (\bar{D}^2 D^\alpha L_\alpha + 4\partial_\mu \Omega^\mu + \text{h.c.}) + \dots, \quad (4.27)$$

where the ellipsis denotes field-dependent terms. We have used that the Weyl weight of Ω is 2. Since the above field-independent part of $\delta_{\text{sc}} \tilde{\Omega}$ is not affected by including higher order corrections in the SUGRA fields, the variation (4.26) becomes

$$\delta_{\text{sc}} S_D[\Omega] = 2 \int d^4x \int d^4\theta \frac{\Omega_0}{3} \left\{ \delta_{\text{sc}} \tilde{E}_2 - \frac{1}{12} \tilde{E}_1 (\bar{D}^2 D^\alpha L_\alpha + 4\partial_\mu \Omega^\mu + \text{h.c.}) \right\}. \quad (4.28)$$

After some calculations, we can show that

$$\delta_{\text{sc}} \left\{ -\frac{1}{8} U_\mu D^\alpha \bar{D}^2 D_\alpha U^\mu + \frac{1}{3} \tilde{E}_1^2 - (\partial_\mu U^\mu)^2 \right\} = \frac{1}{6} \tilde{E}_1 (\bar{D}^2 D^\alpha L_\alpha + 4\partial_\mu \Omega^\mu + \text{h.c.}), \quad (4.29)$$

where total derivatives are dropped. Therefore, \tilde{E}_2 is identified as

$$\tilde{E}_2 = -\frac{1}{16} U_\mu D^\alpha \bar{D}^2 D_\alpha U^\mu + \frac{1}{6} \tilde{E}_1^2 - \frac{1}{2} (\partial_\mu U^\mu)^2. \quad (4.30)$$

This has a similar form to the counterpart of Ref. [9]. The first term of the second line in (4.22) with (4.30) is the kinetic terms for U^μ .

Finally we comment on the relation of the superfield action (4.22) to the component expression (4.19). In order to reproduce the quadratic part of the SUSY Einstein-Hilbert terms $\mathcal{L}_{\text{quad}}^{\text{SG}}$ in (B.3), we also need to count the SUGRA fields contained in the redefined

superfields, in addition to the kinetic terms for U^μ . By including higher order corrections in the SUGRA fields, the redefinition of a real general superfield in (4.13) is extended as

$$\hat{\Omega} \equiv (1 + Y_1 + Y_2) \Omega = (1 + Y_1 + Y_2) \left(\Omega_0 + \tilde{\Omega} \right), \quad (4.31)$$

where $Y_1 \equiv \frac{1}{3} \left(\tilde{\mathcal{E}} + \bar{\tilde{\mathcal{E}}} \right)$ and Y_2 is quadratic in the SUGRA fields. Thus, from (4.22) and (4.31), the Ω_0 -dependent part of the action is expressed as

$$S_D[\Omega] = 2 \int d^4x \int d^4\theta \left(\frac{1}{3} \tilde{E}_2 + \frac{1}{3} \tilde{E}_1 Y_1 + Y_2 \right) \Omega_0 + \dots, \quad (4.32)$$

where the ellipsis denotes terms beyond quadratic order in the SUGRA fields or depending on the matter fields. This corresponds to $\mathcal{L}_{\text{quad}}^{\text{SG}}$ in (B.3).

5 Summary

We have modified the 4D $N = 1$ linearized SUGRA, and provided a complete identification of component fields in each superfield with fields in the superconformal formulation of SUGRA developed in Ref. [4]. The results of our work makes it possible to use both formulations in a complementary manner.

In our modified linearized SUGRA, (anti-) chiral superfields and real general superfields should be understood as the redefined ones defined in (4.13) whose components are identified with the fields in Ref. [4] through (3.12), (3.19), (3.20) and (4.5). The components of the connection superfield U^μ are identified with the SUGRA fields in the Weyl-multiplet as (3.10). The invariant action formulae are expressed in terms of the redefined superfields as

$$\begin{aligned} S_F[W] &= \int d^4x \int d^2\theta \hat{W} + \text{h.c.}, \\ S_D[\Omega] &= 2 \int d^4x \int d^4\theta \left\{ \frac{\Omega_0}{3} \tilde{E}_2 + \left(1 + \frac{1}{3} \tilde{E}_1 \right) \hat{\Omega} \right\}, \\ S_{\text{kin}}^{\text{gauge}}[V] &= \int d^4x \int d^2\theta \left(-\frac{1}{4} \hat{\mathcal{W}}^\alpha \hat{\mathcal{W}}_\alpha \right) + \text{h.c.}, \end{aligned} \quad (5.1)$$

where \tilde{E}_1 and \tilde{E}_2 are defined in (4.8) and (4.30), and the field strength superfield $\hat{\mathcal{W}}_\alpha$ is defined in (4.18). Ω_0 is a constant part of $\hat{\Omega}$, which is set to $-3/2$ for the canonically normalized SUGRA kinetic terms. In the D -term action formula, a chiral multiplet $\hat{\Phi}$ must

be embedded into a general multiplet. Such embedding is provided at the linearized order in the SUGRA fields by

$$\mathcal{V}(\hat{\Phi}) \equiv (1 + iU^\mu \partial_\mu) \hat{\Phi}. \quad (5.2)$$

The gauge transformation of the gauge multiplet \hat{V} is given by

$$\hat{V} \rightarrow \hat{V} + \mathcal{V}(\hat{\Sigma}) + \mathcal{V}(\bar{\hat{\Sigma}}), \quad (5.3)$$

where $\hat{\Sigma}$ is a chiral superfield. The field strength superfield $\hat{\mathcal{W}}_\alpha$ is invariant under this transformation.

Our work will also be useful to discuss higher-dimensional SUGRA. When we consider it in the context of the brane-world scenario, it is convenient to express the action in terms of $N = 1$ superfields, keeping only $N = 1$ SUSY that remains unbroken at low energies manifest. The authors of Ref. [9] construct minimal version of 5D linearized SUGRA along this direction. Although their formulation is powerful to calculate SUGRA loop contributions and is self-consistent, it is not clear how the component fields are related to fields in other off-shell formulations of 5D SUGRA. Especially, it is obscure how to extend their result to more general 5D SUGRA. The superconformal formulation of 5D SUGRA has been developed in Refs. [5, 6], and its $N = 1$ superfield description was provided in Ref. [12, 13]. Hence, by combining these results with our current work, it is possible to obtain the linearized SUGRA formulation of *general* 5D SUGRA. Furthermore, our work is a good starting point to construct an $N = 1$ description of 6D or higher-dimensional SUGRA because the linearized SUGRA formulation is based on the ordinary superspace and thus is easier to handle than the full supergravity. These issues are left for our future works.

Acknowledgements

This work was supported in part by Grant-in-Aid for Young Scientists (B) No.22740187 from Japan Society for the Promotion of Science.

A Component expressions

Here we collect component expressions of some superfields appearing in the text.

The coefficient superfields of the differential operators in (2.13) and (2.14) are written as

$$\begin{aligned}
-\frac{1}{4}\bar{D}^2 L^\alpha &= \epsilon^\alpha + \frac{1}{2}\theta^\alpha \tilde{\varphi} - \frac{1}{2}(\theta\sigma^{\mu\nu})^\alpha \tilde{\lambda}_{\mu\nu} - 2\theta^2 \tilde{\rho}^\alpha - i(\theta\sigma^\mu \bar{\theta}) \partial_\mu \epsilon^\alpha \\
&\quad - \frac{i}{4}\theta^2 \left\{ (\bar{\theta}\bar{\sigma}^\mu)^\alpha \partial_\mu \tilde{\varphi} - (\bar{\theta}\bar{\sigma}^\rho \sigma^{\mu\nu})^\alpha \partial_\rho \tilde{\lambda}_{\mu\nu} \right\} - \frac{1}{4}\theta^2 \bar{\theta}^2 \square_4 \epsilon^\alpha, \\
i\sigma_{\alpha\dot{\alpha}}^\mu \bar{D}^{\dot{\alpha}} L^\alpha + \Omega^\mu &= -\tilde{\xi}_I^\mu + 2i\theta\sigma^\mu \bar{\epsilon} + 2i\bar{\theta}\bar{\sigma}^\mu \epsilon - i(\theta\sigma_\nu \bar{\theta}) \left(\eta^{\mu\nu} \tilde{\varphi} + \tilde{\lambda}^{\mu\nu} - \partial^\nu \tilde{\xi}_I^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\tau} \tilde{\lambda}_{\rho\tau} \right) \\
&\quad - 4i\theta^2 \left\{ \bar{\theta}\bar{\sigma}^\mu \tilde{\rho} + \frac{i}{4}(\bar{\theta}\bar{\sigma}^\nu \sigma^\mu \partial_\nu \bar{\epsilon}) \right\} - \bar{\theta}^2 (\theta\sigma^\nu \bar{\sigma}^\mu \partial_\nu \epsilon) \\
&\quad - \frac{1}{2}\theta^2 \bar{\theta}^2 \partial_\nu \left(\eta^{\mu\nu} \tilde{\varphi} + \tilde{\lambda}^{\mu\nu} - \frac{1}{2}\partial^\nu \tilde{\xi}_I^\mu + \frac{i}{2}\epsilon^{\mu\nu\rho\tau} \tilde{\lambda}_{\rho\tau} \right), \tag{A.1}
\end{aligned}$$

where $\tilde{\varphi} \equiv \tilde{\varphi}_R + i\varphi_I$, $\tilde{\lambda}_{\mu\nu}$, $\tilde{\xi}_I$ and $\tilde{\rho}^\alpha$ are defined in (3.8). An explicit expression of Λ defined in (3.13) is

$$\begin{aligned}
\Lambda &= -\frac{1}{24}(\bar{D}^2 D^\alpha L_\alpha + 4\Xi) \\
&= \frac{1}{2} \left(\tilde{\varphi}_R - \frac{i}{3}\varphi_I \right) - 2\theta\tilde{\rho} - \frac{i}{2}(\theta\sigma^\mu \bar{\theta}) \partial_\mu \left(\tilde{\varphi}_R - \frac{i}{3}\varphi_I \right) \\
&\quad + i\theta^2 (\bar{\theta}\bar{\sigma}^\mu \partial_\mu \tilde{\rho}) - \frac{1}{8}\theta^2 \bar{\theta}^2 \square_4 \left(\tilde{\varphi}_R - \frac{i}{3}\varphi_I \right). \tag{A.2}
\end{aligned}$$

We have taken the gauge (3.4).

For a real general multiplet $[C, \zeta_\alpha, \mathcal{H}, B_\mu, \lambda_\alpha, D]$, each component is embedded into a real superfield that transforms by a law (2.14) as

$$\begin{aligned}
V &= C + i\theta\zeta - i\bar{\theta}\bar{\zeta} - \theta^2 \mathcal{H} - \bar{\theta}^2 \bar{\mathcal{H}} - (\theta\sigma^\mu \bar{\theta}) \left(B_\mu - \zeta\psi_\mu - \bar{\zeta}\bar{\psi}_\mu - \frac{w}{2}CA_\mu \right) \\
&\quad + i\theta^2 \bar{\theta} \left\{ \bar{\lambda} - \frac{i}{2}\bar{\sigma}^\mu (e^{-1})_\mu{}^\nu \partial_\nu \zeta - (\bar{\sigma}^\mu \sigma^\nu \bar{\psi}_\mu) B_\nu + \frac{w}{4}(\bar{\sigma}^\mu \zeta) A_\mu \right\} \\
&\quad - i\bar{\theta}^2 \theta \left\{ \lambda - \frac{i}{2}\sigma^\mu (e^{-1})_\mu{}^\nu \partial_\nu \bar{\zeta} - (\sigma^\mu \bar{\sigma}^\nu \psi_\mu) B_\nu - \frac{w}{4}(\sigma^\mu \bar{\zeta}) A_\mu \right\} \\
&\quad + \frac{1}{2}\theta^2 \bar{\theta}^2 \left\{ D - \frac{1}{2}g^{\mu\nu} \partial_\mu \partial_\nu C - \left(\frac{w-3}{2}A^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\tau} \partial_\nu \tilde{e}_{\rho\tau} \right) B_\mu \right. \\
&\quad \left. + \left(-\bar{\lambda}\bar{\sigma}^\mu \psi_\mu - 2i\partial_\mu \zeta \sigma^{\mu\nu} \psi_\nu + i\partial_\mu \zeta \psi^\mu + \frac{2iw}{3}\zeta \sigma^{\mu\nu} \partial_\nu \psi_\mu + \text{h.c.} \right) \right\}. \tag{A.3}
\end{aligned}$$

The real superfield \tilde{E}_1 defined in (4.8) is expressed as

$$\begin{aligned}
\tilde{E}_1 &= \frac{1}{4} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} [D_\alpha, \bar{D}_{\dot{\alpha}}] U^\mu \\
&= \tilde{e}_\mu{}^\mu + \theta \sigma^\mu \tilde{\psi}_\mu - \bar{\theta} \bar{\sigma}^\mu \tilde{\psi}_\mu + (\theta \sigma^\mu \bar{\theta}) (2d_\mu - \epsilon_{\mu\nu\rho\tau} \partial^\nu \tilde{e}^{\rho\tau}) \\
&\quad + \frac{i}{2} \bar{\theta}^2 \left(\theta \sigma^\mu \bar{\sigma}^\nu \partial_\nu \tilde{\psi}_\mu \right) - \frac{i}{2} \theta^2 \left(\bar{\theta} \bar{\sigma}^\mu \sigma^\nu \partial_\nu \tilde{\psi}_\mu \right) + \frac{1}{4} \theta^2 \bar{\theta}^2 (\square_4 \tilde{e}_\mu{}^\mu - 2\partial_\mu \partial_\nu \tilde{e}^{\mu\nu}) \\
&= \tilde{e}_\mu{}^\mu + 2i (\theta \sigma^\mu \bar{\psi}_\mu + \bar{\theta} \bar{\sigma}^\mu \psi_\mu) + \frac{3}{2} (\theta \sigma^\mu \bar{\theta}) (A_\mu - \epsilon_{\mu\nu\rho\tau} \partial^\nu \tilde{e}^{\rho\tau}) \\
&\quad - 2\bar{\theta}^2 (\theta \partial_\mu \psi^\mu) - 2\theta^2 (\bar{\theta} \partial_\mu \bar{\psi}^\mu) + \frac{1}{4} \theta^2 \bar{\theta}^2 (\square_4 \tilde{e}_\mu{}^\mu - 2\partial_\mu \partial_\nu \tilde{e}^{\mu\nu}). \tag{A.4}
\end{aligned}$$

Thus the density superfield is calculated as

$$\begin{aligned}
1 + \frac{1}{3} (\tilde{E}_1 + \tilde{\mathcal{E}} + \bar{\tilde{\mathcal{E}}}) &= (1 + \tilde{e}_\mu{}^\mu) + \frac{1}{2} (\theta \sigma^\mu \bar{\theta}) (A_\mu - \epsilon_{\mu\nu\rho\tau} \partial^\nu \tilde{e}^{\rho\tau}) \\
&\quad - \frac{1}{3} \bar{\theta}^2 \{ \theta (2\eta^{\mu\nu} + \sigma^\mu \bar{\sigma}^\nu) \partial_\mu \psi_\nu \} - \frac{1}{3} \theta^2 \{ \bar{\theta} (2\eta^{\mu\nu} + \bar{\sigma}^\mu \sigma^\nu) \partial_\mu \bar{\psi}_\nu \} \\
&\quad - \frac{1}{12} \theta^2 \bar{\theta}^2 (\square_4 \tilde{e}_\mu{}^\mu + 2\partial_\mu \partial_\nu \tilde{e}^{\mu\nu}). \tag{A.5}
\end{aligned}$$

B Invariant action formulae

Here we collect the invariant action formulae in Ref. [4] in our notations. For a chiral multiplet $\Phi = [\phi, \chi_\alpha, F]$ with weight $(w, n) = (3, 3)$, the F -term action formula is given by

$$\begin{aligned}
S_F[\Phi] &= \int d^4x \, e (F - i\bar{\psi}_\mu \bar{\sigma}^\mu \chi + \text{h.c.} + \dots) \\
&= \int d^4x \{ (1 + \tilde{e}_\mu{}^\mu) F - i\bar{\psi}_\mu \bar{\sigma}^\mu \chi + \text{h.c.} + \dots \}, \tag{B.1}
\end{aligned}$$

where $e \equiv \det(e_\mu{}^\nu)$, and the ellipsis denotes terms beyond the linear order in the SUGRA fields.

For a real general multiplet $\Omega = [C, \zeta_\alpha, \mathcal{H}, B_\mu, \lambda_\alpha, D]$ with weights $(w, n) = (2, 0)$, the D -term action formula is given by

$$\begin{aligned}
S_D[\Omega] &= \int d^4x \, e \left[D - \bar{\psi}_\mu \bar{\sigma}^\mu \lambda + \psi_\mu \sigma^\mu \bar{\lambda} + \frac{4i}{3} (\zeta \sigma^{\mu\nu} \partial_\mu \psi_\nu - \bar{\zeta} \bar{\sigma}^{\mu\nu} \partial_\mu \bar{\psi}_\nu) \right. \\
&\quad \left. + \frac{C}{3} \{ R(\omega) + 4\epsilon^{\mu\nu\rho\tau} (\psi_\mu \sigma_\tau \partial_\nu \bar{\psi}_\rho - \bar{\psi}_\mu \bar{\sigma}_\tau \partial_\nu \psi_\rho) \} + \dots \right] \\
&= \int d^4x \left[(1 + \tilde{e}_\mu{}^\mu) D + \left(\psi_\mu \sigma^\mu \bar{\lambda} + \frac{4i}{3} \zeta \sigma^{\mu\nu} \partial_\mu \psi_\nu + \text{h.c.} \right) \right. \\
&\quad \left. + \frac{2\tilde{C}}{3} (\partial^\mu \partial_\nu \tilde{e}_\mu{}^\nu - \square_4 \tilde{e}_\mu{}^\mu) + \mathcal{L}_{\text{quad}}^{\text{SG}} + \dots \right], \tag{B.2}
\end{aligned}$$

where $R(\omega)$ is the scalar curvature constructed from the spin connection, $\tilde{C} \equiv C - \Omega_0$ where Ω_0 is a constant to which C will be set by the \mathbf{D} -gauge fixing. Thus \tilde{C} will vanish after the \mathbf{D} -gauge fixing. The quadratic part in the SUGRA fields $\mathcal{L}_{\text{quad}}^{\text{SG}}$ is given by

$$\begin{aligned} \mathcal{L}_{\text{quad}}^{\text{SG}} = \frac{\Omega_0}{3} \{ & \tilde{e}_\mu{}^\mu (2\partial^\nu \partial_\rho \tilde{e}_\nu{}^\rho - \square_4 \tilde{e}_\nu{}^\nu) - \tilde{e}^{\mu\nu} (\partial_\nu \partial^\rho \tilde{e}_{(\mu\rho)} + \partial_\mu \partial^\rho \tilde{e}_{(\nu\rho)} - \square_4 \tilde{e}_{(\mu\nu)}) \\ & + 4\epsilon^{\mu\nu\rho\tau} (\psi_\mu \sigma_\tau \partial_\nu \bar{\psi}_\rho + \text{h.c.}) \} , \end{aligned} \quad (\text{B.3})$$

where $\tilde{e}_{(\mu\nu)} \equiv \frac{1}{2} (\tilde{e}_{\mu\nu} + \tilde{e}_{\nu\mu})$, and the ellipsis denotes terms beyond the linear order in the SUGRA fields.

References

- [1] M. Kaku, P. K. Townsend, P. van Nieuwenhuizen, Phys. Rev. **D17** (1978) 3179.
- [2] M. Kaku, P. K. Townsend, Phys. Lett. **B76** (1978) 54.
- [3] S. Ferrara, M. T. Grisaru, P. van Nieuwenhuizen, Nucl. Phys. **B138** (1978) 430.
- [4] T. Kugo, S. Uehara, Nucl. Phys. **B226** (1983) 49.
- [5] T. Kugo, K. Ohashi, Prog. Theor. Phys. **105** (2001) 323-353. [hep-ph/0010288]; T. Fujita, T. Kugo, K. Ohashi, Prog. Theor. Phys. **106** (2001) 671-690 [hep-th/0106051].
- [6] T. Kugo, K. Ohashi, Prog. Theor. Phys. **108** (2002) 203-228. [hep-th/0203276].
- [7] S. Ferrara, B. Zumino, Nucl. Phys. **B134** (1978) 301.
- [8] W. Siegel, S. J. Gates, Jr., Nucl. Phys. **B147** (1979) 77.
- [9] W. D. Linch, III, M. A. Luty, J. Phillips, Phys. Rev. **D68** (2003) 025008. [hep-th/0209060].
- [10] I. L. Buchbinder, S. J. Gates, Jr., H. -S. Goh, W. D. Linch, III, M. A. Luty, S. -P. Ng, J. Phillips, Phys. Rev. **D70** (2004) 025008. [hep-th/0305169].
- [11] J. Wess, J. Bagger, “Supersymmetry and supergravity,” Princeton, USA: Univ. Pr. (1992) 259 p.
- [12] F. Paccetti Correia, M. G. Schmidt, Z. Tavartkiladze, , Nucl. Phys. **B709** (2005) 141-170. [hep-th/0408138].
- [13] H. Abe, Y. Sakamura, JHEP **0410** (2004) 013. [hep-th/0408224].